

## 2. Klausur

für Studierende der Fachrichtungen  
el, geod, kyb

**Bitte unbedingt beachten:**

- In dieser Klausur können bis zu **46 Punkte** erreicht werden.

**Aufgabe 1** (9 Punkte): Parametrisieren Sie folgende Fläche:

(a)  $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 6y^2 = z, z \in [0, h]\}, h \in \mathbb{R}, h > 0.$

$$\omega(\varphi, r) = \begin{pmatrix} \sqrt{r} \cos \varphi \\ \sqrt{\frac{r}{6}} \sin \varphi \\ r \end{pmatrix} \quad \varphi \in [0, 2\pi), r \in [0, h].$$

(b) Ferner gilt:

$$\omega_\varphi(\varphi, r) = \begin{pmatrix} -\sqrt{r} \sin \varphi \\ \sqrt{\frac{r}{6}} \cos \varphi \\ 0 \end{pmatrix}$$
$$\omega_r(\varphi, r) = \begin{pmatrix} \frac{1}{2\sqrt{r}} \cos \varphi \\ \frac{1}{2\sqrt{6r}} \sin \varphi \\ 1 \end{pmatrix}.$$

Es folgt dann:

$$\begin{aligned} \omega_\varphi \times \omega_r &= \begin{pmatrix} \omega_{\varphi,2}\omega_{r,3} - \omega_{\varphi,3}\omega_{r,2} \\ \omega_{\varphi,3}\omega_{r,1} - \omega_{\varphi,1}\omega_{r,3} \\ \omega_{\varphi,1}\omega_{r,2} - \omega_{\varphi,2}\omega_{r,1} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{r}{6}} \cos \varphi \\ \sqrt{r} \sin \varphi \\ -\sqrt{r} \sin \varphi \frac{1}{2\sqrt{6r}} \sin \varphi - \sqrt{\frac{r}{6}} \cos \varphi \frac{1}{2\sqrt{r}} \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{r}{6}} \cos \varphi \\ \sqrt{r} \sin \varphi \\ -\frac{1}{2\sqrt{6}} \end{pmatrix}. \end{aligned}$$

Andererseits gilt:  $\omega(\frac{\pi}{2}, \frac{h}{2}) = (0, \sqrt{\frac{h}{12}}, \frac{h}{2})$ . Also folgt:

$$\begin{aligned} n \left( 0, \sqrt{\frac{h}{12}}, \frac{h}{2} \right) &= \frac{\omega_\varphi \times \omega_r(\frac{\pi}{2}, \frac{h}{2})}{|\omega_\varphi \times \omega_r(\frac{\pi}{2}, \frac{h}{2})|} \\ &= \sqrt{\frac{24}{12h+1}} \begin{pmatrix} 0 \\ \sqrt{\frac{h}{2}} \\ -\frac{1}{2\sqrt{6}} \end{pmatrix} = \frac{2\sqrt{6}}{\sqrt{12h+1}} \begin{pmatrix} 0 \\ \sqrt{\frac{h}{2}} \\ -\frac{1}{2\sqrt{6}} \end{pmatrix}. \end{aligned}$$

**Aufgabe 2** (10 Punkte): Es gilt

$$\frac{1}{4+x^4} = \frac{1}{(x^2+2i)(x^2-2i)} = \frac{1}{(x-1-i)(x+1-i)(x-1+i)(x+1+i)}.$$

Da der Grad des Nenners  $n = 4$  und der Grad des Zählers  $m = 0$  ist, gilt  $n \geq m + 2$ . Aus der Vortragsübung ist bekannt, dass daher

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{4+x^4} &= 2\pi i \operatorname{Res} \left( \frac{1}{4+z^4}, 1+i \right) + 2\pi i \operatorname{Res} \left( \frac{1}{4+z^4}, -1+i \right) \\ &= 2\pi i \lim_{z \rightarrow 1+i} \frac{1}{(z+1-i)(z-1+i)(z+1+i)} \\ &\quad + 2\pi i \lim_{z \rightarrow -1+i} \frac{1}{(z-1-i)(z-1+i)(z+1+i)} \\ &= \frac{2\pi i}{(1+i+1-i)(1+i-1+i)(1+i+1+i)} \\ &\quad + \frac{2\pi i}{(-1+i-1-i)(-1+i-1+i)(-1+i+1+i)} \\ &= \frac{2\pi i}{2(2i)(2+2i)} + \frac{2\pi i}{(-2)(-2+2i)(2i)} \\ &= \frac{\pi}{4+4i} + \frac{\pi}{4-4i} \\ &= \frac{\pi(4-4i) + \pi(4+4i)}{16+16} \\ &= \frac{8\pi}{32} = \frac{\pi}{4}. \end{aligned}$$

**Aufgabe 3** (8 Punkte):

Bestimmen Sie die Fourierkoeffizienten der folgenden Funktion in komplexer und in Sinus-Cosinus-Form.

$$f(x) = 3|x - 2\pi l| + 2 \text{ für } x \in [(2l - 1)\pi, (2l + 1)\pi], l \in \mathbb{Z},$$

Es gilt:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k = 0, 1, 2, \dots$$

und

$$c_0 = \frac{1}{2} a_0 \quad c_k = \frac{1}{2} (a_k - i b_k) \quad c_{-k} = \frac{1}{2} (a_k + i b_k).$$

Also folgt

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (3|x| + 2) dx = 3\pi + 4.$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} (3|x| + 2) \cos(kx) dx \\ &= \frac{1}{\pi} \left( \int_0^{\pi} 3x \cos(kx) dx - \int_{-\pi}^0 3x \cos(kx) dx \right) \\ &= \frac{1}{\pi} \left( \left[ \frac{3}{k} x \sin(kx) \right]_0^{\pi} - \int_0^{\pi} \frac{3}{k} \sin(kx) dx - \left[ \frac{3}{k} x \sin(kx) \right]_{-\pi}^0 + \int_{-\pi}^0 \frac{3}{k} \sin(kx) dx \right) \\ &= \frac{1}{\pi} \left( \frac{3}{k^2} [\cos(kx)]_0^{\pi} - \frac{3}{k^2} [\cos(kx)]_{-\pi}^0 \right) \\ &= \frac{6}{\pi k^2} ((-1)^k - 1). \end{aligned}$$

$$b_k = 0, \text{ da } f \text{ gerade. } c_0 = \frac{3\pi}{2} + 2, c_{\pm k} = \frac{3}{\pi k^2} ((-1)^k - 1).$$

**Aufgabe 4** (7 Punkte):

$$2(x+1)e^{2y}dy + (e^{2y} - 2x)dx = 0, \quad y(0) = 0, \quad x > 0.$$

$$\begin{aligned} P(y, x) &= e^{2y} - 2x & \frac{\partial P}{\partial y} &= 2e^{2y} \\ Q(y, x) &= 2e^{2y}(x+1) & \frac{\partial Q}{\partial x} &= 2e^{2y} \end{aligned}$$

Suche nach  $U(y, x) \in C^1(\mathbb{R} \times \mathbb{R}_+)$  mit

$$\nabla U(y, x) = \begin{pmatrix} Q(y, x) \\ P(y, x) \end{pmatrix}$$

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = 0 \iff U(y, x) = C \text{ (Konstante)}$$

$$U(y, x) = \int_{x_0}^x P(y_0, s) ds + \int_{y_0}^y Q(\zeta, x) d\zeta,$$

mit  $(y_0, x_0) \in \mathbb{R} \times \mathbb{R}_+$  fest und  $(y, x) \in \mathbb{R} \times \mathbb{R}_+$ .

$$U(y, x) = \int_{x_0}^x (e^{2y_0} - 2s) ds + \int_{y_0}^y 2e^{2\zeta}(x+1) d\zeta.$$

Wähle  $y_0 = 0, x_0 = 0$

$$\begin{aligned} U(y, x) &= \int_0^x (1 - 2s) ds + \int_0^y 2e^{2\zeta}(x+1) d\zeta \\ &= x - x^2 + (x+1) \cdot (e^{2y} - 1), \end{aligned}$$

also ist

$$C = (x+1)e^{2y} - (x+1) - x^2 + x.$$

$$\iff (x+1)e^{2y} = C + x^2 + 1$$

$$\iff e^{2y} = \frac{x^2 + 1 + C}{x+1}$$

$$\implies y(x) = \frac{1}{2} \ln \frac{x^2 + 1 + C}{x+1} \quad y: \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$0 = y(0) = \frac{1}{2} \ln(C+1) \iff C = 0$$

$$\implies y(x) = \frac{1}{2} \ln \frac{x^2 + 1}{x+1}.$$

Maximaler Definitionsbereich ist  $\mathbb{R}_+$ .

**Aufgabe 5** (11 Punkte):

$$v(x) = h(x_1 + x_2 + x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Delta v_1 + \Delta v_2 + \Delta v_3 = 0.$$

Substitution:  $u := x_1 + x_2 + x_3$ .

(a)

$$\begin{aligned} \frac{\partial v_1}{\partial x_1} &= h'(u)x_1 + h(u) \\ \frac{\partial^2 v_1}{\partial x_1^2} &= h''(u)x_1 + 2h'(u) \\ \frac{\partial v_1}{\partial x_2} &= h'(u)x_1 \\ \frac{\partial^2 v_1}{\partial x_2^2} &= h''(u)x_1 \\ \frac{\partial^2 v_1}{\partial x_3^2} &= h''(u)x_1. \end{aligned}$$

$$\begin{aligned} \implies \Delta v_1 &= 3h''(u)x_1 + 2h'(u) \\ \Delta v_2 &= 3h''(u)x_2 + 2h'(u) \\ \Delta v_3 &= 3h''(u)x_3 + 2h'(u). \end{aligned}$$

(b)

$$\Delta v_1 + \Delta v_2 + \Delta v_3 = 3h''(u)u + 6h'(u).$$

$$\implies h''(u) = -\frac{2}{u}h'(u), \quad h(1) = 1, \quad h'(1) = 1.$$

Substitution  $h'(u) = y(u)$ .

$$\implies \dot{y}(u) = -\frac{2}{u}y(u) \quad y(1) = 1.$$

Trennung der Variablen:

$$\begin{aligned} g(u) &= -\frac{2}{u} \implies G(u) = -2 \ln u \\ H(y) &= \frac{1}{y} \implies H(y) = \int_1^y \frac{1}{\zeta} d\zeta = \ln y \end{aligned}$$

$$H : \mathbb{R}_+ \rightarrow \mathbb{R} \quad H^{-1} : \mathbb{R} \rightarrow \mathbb{R}_+ \quad H^{-1}(y) = e^y$$

$$\implies y(u) = \mathcal{H}^{-1}(-2 \ln u) = e^{-2 \ln u} = \frac{1}{u^2}, \quad y : \mathbb{R} \rightarrow \mathbb{R}_+$$

$$\implies h'(u) = \frac{1}{u^2}, \quad h(1) = 1.$$

$$\implies h(u) = 2 - \frac{1}{u}, \quad h : [1, \infty) \rightarrow \mathbb{R}.$$